MATHEMATICS OF COMPUTATION
VOLUME 64, NUMBER 210
APRIL 1995, PAGES S7–S11

Supplement to
FINITE DIFFERENCE METHOD FOR GENERALIZED ZAKHAROV EQUATIONS

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In this section, long and highly technical proofs of two Lemmas in Section 3 are given.

Proof of Lemma 4. Direct computation implies that

$$
\begin{aligned}
&P_1^{n+\frac12} - P_2^{n+\frac12}\\
=&\mathrm{Re}\Bigg\{h\sum_{j=1}^{J}\Bigg[(N(j,n)+N(j,n+1))(F(|E(j,n+1)|^2)+F(|E(j,n)|^2))\\
&-(N(j,n)+N(j,n+1))\frac{F(|E(j,n+1)|^2)-F(|E(j,n)|^2)}{|E(j,n+1)|^2-|E(j,n)|^2}(E(j,n+1)+E(j,n))\\
&\cdot(\overline{E_j^{n+1}-E_j^n}) - (N_j^n+N_j^{n+1})\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}(E_j^{n+1}+E_j^n)\\
&\cdot(\overline{E(j,n+1)-E(j,n)}) + (N_j^n+N_j^{n+1})(F(|E_j^{n+1}|^2)-F(|E_j^n|^2))\Bigg]\Bigg\}\\
&-h\sum_{j=1}^{J}[F(|E(j,n+1)|^2)-F(|E(j,n)|^2)-F(|E_j^{n+1}|^2)+F(|E_j^n)|^2)]\\
&\cdot[N(j,n+1)+N(j,n)-N_j^{n+1}-N_j^n]\\
=&\mathrm{Re}\Bigg\{h\sum_{j=1}^{J}[(N_j^{n+1}+N_j^n)(\overline{E(j,n+1)-E(j,n)})-(N(j,n)+N(j,n+1))\\
&\cdot(\overline{E_j^{n+1}-E_j^n})]\Bigg[\frac{F(|E(j,n+1)|^2-F(|E(j,n)|^2)}{|E(j,n+1)|^2-|E(j,n)|^2}(E(j,n+1)+E(j,n))\\
&-\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}(E_j^{n+1}+E_j^n)\Bigg]\Bigg\}\\
=&\mathrm{Re}\Bigg\{h\sum_{j=1}^{J}[(N(j,n)+N(j,n+1))(\overline{e_j^{n+1}-e_j^n})-(\overline{E(j,n+1)-E(j,n)}\\
&\cdot(\eta_j^{n+1}+\eta_j^n)]\Bigg[\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}(e_j^{n+1}+e_j^n)
\end{aligned}
$$

$$+\left(\frac{F(|E(j,n+1)|^2-F(|E(j,n)|^2)}{|E(j,n+1)|^2-|E(j,n)|^2}-\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}\right)$$

$$\cdot(E(j,n+1)+E(j,n))\Big]\Big\}. \tag{4.1}$$

Making Taylor's expansion we have

$$\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}=\sum_{l=1}^{\infty}\frac{1}{l!}F^{(l)}(|E_j^n|^2)(|E_j^{n+1}|^2-|E_j^n|^2)^{l-1}. \tag{4.2}$$

Using the formulae

$$a^l-b^l=(a-b)\sum_{k=0}^{l-1}a^{l-1-k}\cdot b^k \quad , \quad (a-b)^l=\sum_{k=0}^{l}(-1)^l C_l^k a^{l-k}\cdot b^k,$$

the estimates in Lemma 3 and $E(x,t)\in C^{(5)}, N(x,t)\in C^{(5)}$, we obtain

$$\left|\frac{F(|E(j,n+1)|^2-F(|E(j,n)|^2)}{|E(j,n+1)|^2-|E(j,n)|^2}-\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}\right|$$

$$=\Bigg|\sum_{l=1}^{\infty}\frac{1}{l!}[F^{(l)}(|E(j,n)|^2)((|E(j,n+1)|^2-|E(j,n)|^2)^{l-1}-(|E_j^{n+1}|^2-|E_j^n|^2)^{l-1})$$

$$+(F^{(l)}(|E(j,n)|^2)-F^{(l)}(|E_j^n|^2))(|E_j^{n+1}|^2-|E_j^n|^2)^{l-1}]\Bigg|$$

$$=\Bigg|\sum_{l=1}^{\infty}\frac{1}{l!}[F^{(l)}(|E(j,n)|^2)(|E(j,n+1)|^2-|E(j,n)|^2-|E_j^{n+1}|^2+|E_j^n|^2)$$

$$\cdot\sum_{k=0}^{l-2}(|E(j,n+1)|^2-|E(j,n)|^2)^{l-2-k}(|E_j^{n+1}|^2-|E_j^n|^2)^k$$

$$+F^{(l+1)}(\xi_1)(|E(j,n)|^2-|E_j^n|^2)(|E_j^{n+1}|^2-|E_j^n|^2)^{l-1}]\Bigg|$$

$$\le C(|e_j^n|+|e_j^{n+1}|), \tag{4.3}$$

where ξ_1 is located between $|E(j,n)|^2$ and $|E_j^n|^2$,

$$\left|\left(\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}\right)_x\right|=\frac{1}{h}\left|\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}-\frac{F(|E_{j-1}^{n+1}|^2)-F(|E_{j-1}^n|^2)}{|E_{j-1}^{n+1}|^2-|E_{j-1}^n|^2}\right|$$

$$\le C(|(E_j^{n+1})_x|+|(E_j^n)_x|), \tag{4.4}$$

and

$$P_x\equiv\left|\left(\frac{F(|E(j,n+1)|^2)-F(|E(j,n)|^2)}{|E(j,n+1)|^2-|E(j,n)|^2}-\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}\right)_x\right|$$

$$=\frac{1}{h}\Bigg|\sum_{l=1}^{\infty}\frac{1}{l!}[F^{(l)}(|E(j,n)|^2)(|E(j,n+1)|^2-|E(j,n)|^2)^{l-1}$$

$$-F^{(l)}(|E(j-1,n)|^2)(|E(j-1,n+1)|^2-|E(j-1,n)|^2)^{l-1}$$

$$-F^{(l)}(|E_j^n)|^2)(|E_j^{n+1}|^2-|E_j^n|^2)^{l-1}+F^{(l)}(|E_{j-1}^n)|^2)(|E_{j-1}^{n+1}|^2-|E_{j-1}^n|^2)^{l-1}\Bigg|$$

$$=\frac{1}{h}\Bigg|\sum_{l=1}^{\infty}\frac{1}{l!}\Bigg\{\sum_{k=0}^{l-1}(-1)^k C_{l-1}^k[F^{(l)}(|E(j,n)|^2)|E(j,n+1)|^{2(l-1-k)}|E(j,n)|^{2k}$$

$$-F^{(l)}(|E(j-1,n)|^2)(|E(j-1,n+1)|^{2(l-1-k)}|E(j-1,n)|^{2k}$$

$$-F^{(l)}(|E_j^n|^2)|E_j^{n+1}|^{2(l-1-k)}|E_j^n|^{2k}+F^{(l)}(|E_{j-1}^n)|^2)|E_{j-1}^{n+1}|^{2(l-1-k)}|E_{j-1}^n|^{2k}]\Bigg\}\Bigg|$$

$$=\frac{1}{h}\Bigg|\sum_{l=1}^{\infty}\frac{1}{l!}\Bigg\{\sum_{k=0}^{l-1}(-1)^k C_{l-1}^k[(F^{(l)}(|E(j,n)|^2)-F^{(l)}(|E_j^n|^2))|E(j,n+1)|^{2(l-1-k)}|E(j,n)|^{2k}$$

$$+F^{(l)}(|E_j^n)|^2)(|E(j,n+1)|^{2(l-1-k)}-|E_j^{n+1}|^{2(l-1-k)})|E(j,n)|^{2k}$$

$$+F^{(l)}(|E_j^n)|^2)|E_j^{n+1}|^{2(l-1-k)}(|E(j,n)|^{2k}-|E_j^n|^{2k})$$

$$-(F^{(l)}(|E(j-1,n)|^2)-F^{(l)}(|E_{j-1}^n|^2))|E(j-1,n+1)|^{2(l-1-k)}|E(j-1,n)|^{2k}$$

$$-F^{(l)}(|E_{j-1}^n|^2)(|E(j-1,n+1)|^{2(l-1-k)}-|E_{j-1}^{n+1}|^{2(l-1-k)})|E(j-1,n)|^{2k}$$

$$-F^{(l)}(|E_{j-1}^n)|^2)|E_{j-1}^{n+1}|^{2(l-1-k)}(|E(j-1,n)|^{2k}-|E_{j-1}^n|^{2k})]\Bigg\}\Bigg|.$$

Making Taylor's expansion , using

$$a^{2l}-b^{2l}=(a^2-b^2)\sum_{m=0}^{l-1}a^{2(l-1-m)}\cdot b^{2m}$$

and

$$|E(j,n)|^2-|E_j^n|^2=\mathrm{Re}(e_j^n\cdot(\overline{E(j,n)+E_j^n}))$$

etc., we have

$$
\begin{aligned}
P_x = \Bigg| \sum_{l=1}^{\infty} \frac{1}{l!} \Bigg\{ \sum_{k=0}^{l-1} (-1)^k C_{l-1}^k \cdot \mathrm{Re} \Bigg[\\
& \sum_{m=1}^{\infty} \frac{1}{m!}\frac{1}{h} \big(F^{(l+m)}(|E_j^n|^2)(e_j^n)^m (\overline{E(j,n)+E_j^n})^m |E(j,n+1)|^{2(l-1-k)} |E(j,n)|^{2k} \\
& - F^{(l+m)}(|E_{j-1}^n|^2)(e_{j-1}^n)^m (\overline{E(j-1,n)+E_{j-1}^n})^m \\
& \cdot |E(j-1,n+1)|^{2(l-1-k)} |E(j-1,n)|^{2k}\big) \\
& + \frac{1}{h}\Bigg(F^{(l)}(|E_j^n|^2)|E(j,n)|^{2k} \overline{e_j^{n+1}}(E(j,n+1)+E_j^{n+1}) \sum_{m=0}^{l-k-2} |E(j,n+1)|^{2(l-k-2-m)} \\
& \cdot |E_j^{n+1}|^{2m} - F^{(l)}(|E_{j-1}^n|^2)|E(j-1,n)|^{2k} \overline{e_{j-1}^{n+1}}(E(j-1,n+1)+E_{j-1}^{n+1}) \\
& \cdot \sum_{m=0}^{l-k-2} |E(j-1,n+1)|^{2(l-k-2-m)} |E_{j-1}^{n+1}|^{2m} \Bigg) \\
& + \frac{1}{h}\Bigg(F^{(l)}(|E_j^n|^2)|E_j^{n-1}|^{2(l-1-k)} \overline{e_j^n}(E(j,n)+E_j^n) \sum_{m=0}^{k-1} |E(j,n)|^{2(k-1-m)} |E_j^n|^{2m} \\
& - F^{(l)}(|E_{j-1}^n|^2)|E_{j-1}^{n+1}|^{2(l-1-k)} \overline{e_{j-1}^n}(E(j-1,n)+E_{j-1}^n) \\
& \cdot \sum_{m=0}^{k-1} |E(j-1,n)|^{2(k-1-m)} |E_{j-1}^n|^{2m} \Bigg) \Bigg] \Bigg\} \Bigg| .
\end{aligned}
\tag{4.5}
$$

Now, we estimate the terms in (4.5):

$$
\begin{aligned}
& \frac{1}{h}\Big| F^{(l+m)}(|E_j^n|^2)(e_j^n)^m (\overline{E(j,n)+E_j^n})^m |E(j,n+1)|^{2(l-1-k)} |E(j,n)|^{2k} - F^{(l+m)} \\
& \cdot (|E_{j-1}^n|^2)(e_{j-1}^n)^m (\overline{E(j-1,n)+E_{j-1}^n})^m |E(j-1,n+1)|^{2(l-1-k)} |E(j-1,n)|^{2k} \Big| \\
= & \frac{1}{h}\Big| \Big\{ [F^{(l+m)}(|E_j^n|^2) - F^{(l+m)}(|E_{j-1}^n|^2)](e_j^n)^m (\overline{E(j,n)+E_j^n})^m \\
& \cdot |E(j,n+1)|^{2(l-1-k)} |E(j,n)|^{2k} \\
& + F^{(l+m)}(|E_{j-1}^n|^2)[(e_j^n)^m - (e_{j-1}^n)^m](\overline{E(j,n)+E_j^n})^m |E(j,n+1)|^{2(l-1-k)} |E(j,n)|^{2k} \\
& + F^{(l+m)}(|E_{j-1}^n|^2)(e_{j-1}^n)^m [(\overline{E(j,n)+E_j^n})^m - (\overline{E(j-1,n)+E_{j-1}^n})^m] \\
& \cdot |E(j,n+1)|^{2(l-1-k)} |E(j,n)|^{2k} | \\
& + F^{(l+m)}(|E_{j-1}^n|^2)(e_{j-1}^n)^m (\overline{E(j-1,n)+E_{j-1}^n})^m \\
& \cdot [|E(j,n+1)|^{2(l-1-k)} - |E(j-1,n+1)|^{2(l-1-k)}] \cdot |E(j,n)|^{2k} | \\
& + F^{(l+m)}(|E_{j-1}^n|^2)(e_{j-1}^n)^m (\overline{E(j-1,n)+E_{j-1}^n})^m \\
& \cdot |E(j-1,n+1)|^{2(l-1-k)} [|E(j,n)|^{2k} - |E(j-1,n)|^{2k}] \Big\} \Big| \\
\le & C|(E_{j-1}^n)_x| \cdot |e_j^n|(2C)^{m-1} \cdot (2C)^m C^{2(l-1)} + C|(e_{j-1}^n)_x|(2C)^{m-1}(2C)^m C^{2(l-1)} \\
& + C(2C)^{m-1} \cdot |e_{j-1}^n|(C + |(E_{j-1}^n)_x|) \cdot 2(2C)^{m-1} \cdot C^{2(l-1)} \\
& + C(2C)^{m-1} \cdot |e_{j-1}^n|(2C)^m C \cdot (2C)^{2(l-1-k)-1} \cdot C^{2k} \\
& + C \cdot (2C)^{m-1} |e_{j-1}^n|(2C)^m C^{2(l-1-k)} \cdot C \cdot (2C)^{2k-1} \\
\le & C(2C)^{2m} \cdot C^{2(l-1)}(|(E_{j-1}^n)_x| \cdot |e_j^n| + |(e_{j-1}^n)_x| + |e_{j-1}^n| \cdot |(E_{j-1}^n)_x| + |e_{j-1}^n|).
\end{aligned}
$$

Other terms in the inequalities (4.5) can be estimated similarly, and substituting these estimates in (4.5) implies that

$$
\begin{aligned}
P_x \le & C(|(e_{j-1}^{n+1})_x| + |(e_{j-1}^n)_x| + |e_j^{n+1}| + |e_{j-1}^{n+1}| + |e_j^n| + |e_{j-1}^n| + \\
& (|e_j^{n+1}| + |e_{j-1}^{n+1}| + |e_j^n| + |e_{j-1}^n|)(|(E_{j-1}^{n+1})_x| + |(E_{j-1}^n)_x|)).
\end{aligned}
\tag{4.6}
$$

Thus, using the inequalities (4.2) and (4.3) we first estimate a simpler term in (4.1):

$$
\begin{aligned}
& \Bigg| \mathrm{Re} \Bigg\{ h \sum_{j=1}^{J} \overline{(E(j,n+1) - E(j,n))}(\eta_j^{n+1} + \eta_j^n) \Bigg[\frac{F(|E_j^{n+1}|^2) - F(|E_j^n|^2)}{|E_j^{n+1}|^2 - |E_j^n|^2}(e_j^{n+1} + e_j^n) \\
& + \left(\frac{F(|E(j,n+1)|^2) - F(|E(j,n)|^2)}{|E(j,n+1)|^2 - |E(j,n)|^2} - \frac{F(|E_j^{n+1}|^2) - F(|E_j^n|^2)}{|E_j^{n+1}|^2 - |E_j^n|^2} \right)(E(j,n+1) + E(j,n)) \Bigg] \Bigg\} \Bigg| \\
\le & \Bigg| h \sum_{j=1}^{J} (\tau \overline{E_t(j,\xi_2)})(\eta_j^{n+1} + \eta_j^n) \Bigg[\frac{F(|E_j^{n+1}|^2) - F(|E_j^n|^2)}{|E_j^{n+1}|^2 - |E_j^n|^2}(e_j^{n+1} - e_j^n) \\
& + \left(\frac{F(|E(j,n+1)|^2) - F(|E(j,n)|^2)}{|E(j,n+1)|^2 - |E(j,n)|^2} - \frac{F(|E_j^{n+1}|^2) - F(|E_j^n|^2)}{|E_j^{n+1}|^2 - |E_j^n|^2} \right)(E(j,n+1) + E(j,n)) \Bigg] \Bigg| \\
\le & C\tau(\|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2 + \|e^{n+1}\|_2^2 + \|e^n\|_2^2),
\end{aligned}
\tag{4.7}
$$

where ξ_2 is located between t^n and t^{n+1}. Then, using the error equation (3.3) and summing by parts, we have

$$
\begin{aligned}
&\left|\mathrm{Re}\left[h\sum_{j=1}^{J}(N(j,n)+N(j,n+1))\overline{(e_j^{n+1}-e_j^n)}\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}(e_j^{n+1}+e_j^n)\right]\right| \\
=\tau&\left|\mathrm{Re}\left\{h\sum_{j=1}^{J}(N(j,n)+N(j,n+1))\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}\overline{(e_j^{n+1}+e_j^n)}\right.\right. \\
&\cdot i[\frac{1}{2}((e_j^{n+1})_{x\bar{x}}+(e_j^n)_{x\bar{x}}-R^E \\
&-\frac{1}{4}(N(j,n)+N(j,n+1))\frac{F(|E(j,n+1)|^2)-F(|E(j,n)|^2)}{|E(j,n+1)|^2-|E(j,n)|^2}(E(j,n+1)+E(j,n)) \\
&\left.\left.+\frac{1}{4}(N_j^n+N_j^{n+1})\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}(E_j^{n+1}+E_j^n)\right]\right| \\
\le\tau&\left|\frac{1}{2}h\sum_{j=1}^{J}\left[(N(j,n)+N(j,n+1))\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}\overline{(e_j^{n+1}+e_j^n)}\right]_x\right. \\
&\left.\cdot((e_j^{n+1})_x+(e_j^n)_x)\right|+C\tau(h^2+\tau^2)^2+C\tau(\|e^{n+1}\|_2^2+\|e^n\|_2^2) \\
&+\tau\left|\frac{1}{4}h\sum_{j=1}^{J}(N(j,n)+N(j,n+1))\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}\overline{(e_j^{n+1}+e_j^n)}\right. \\
&\cdot\left[(\eta_j^n+\eta_j^{n+1})\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}(E_j^{n+1}+E_j^n)+(N(j,n)+N(j,n+1))\right. \\
&\cdot\left(\frac{F(|E(j,n+1)|^2)-F(|E(j,n)|^2)}{|E(j,n+1)|^2-|E(j,n)|^2}-\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}\right)(E_j^{n+1}+E_j^n) \\
&\left.\left.+(N(j,n)+N(j,n+1))\frac{F(|E(j,n+1)|^2)-F(|E(j,n)|^2)}{|E(j,n+1)|^2-|E(j,n)|^2}(e_j^{n+1}+e_j^n)\right]\right|.
\end{aligned}
$$

Furthermore, using inequalities (4.2), (4.3), (4.4) and (4.6), we obtain

$$
\begin{aligned}
&|\mathrm{Re}[h\sum_{j=1}^{J}(N(j,n)+N(j,n+1))\overline{(e_j^{n+1}-e_j^n)}) \\
&\cdot\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}(e_j^{n+1}-e_j^n)]| \\
\le C\tau&\left(\|e_x^{n+1}\|_2^2+\|e_x^n\|_2^2+\|e^{n+1}\|_2^2+\|e^n\|_2^2\right. \\
&\left.+h\sum_{j=1}^{J}|(e_j^{n+1}+e_j^n)((e_j^{n+1})_x+(e_j^n)_x)(|(E_j^{n+1})_x|+|(E_j^n)_x|)|\right) \\
&+C\tau(h^2+\tau)^2+C\tau(\|e^{n+1}\|_2^2+\|e^n\|_2^2) \\
&+C\tau(\|e^{n+1}\|_2^2+\|e^n\|_2^2+\|\eta^{n+1}\|_2^2+\|\eta^n\|_2^2) \\
\le C\tau&(h^2+\tau^2)^2+C\tau(\|e_x^{n+1}\|_2^2+\|e_x^n\|_2^2+\|e^{n+1}\|_2^2+\|e^n\|_2^2 \\
&+\|\eta^{n+1}\|_2^2+\|\eta^n\|_2^2)+C\tau\left[h\sum_{j=1}^{J}(|(e_j^{n+1})_x|^2+|(e_j^n)_x|^2)\right. \\
&\left.+h\sum_{j=1}^{J}(|(E_j^{n+1})_x|^2+|(E_j^n)_x|^2)(\|e^{n+1}\|_\infty^2+\|e^n\|_\infty^2)\right] \\
\le C\tau&(h^2+\tau^2)^2+C\tau(\|e_x^{n+1}\|_2^2+\|e_x^n\|_2^2+\|e^{n+1}\|_2^2+\|e^n\|_2^2 \\
&+\|\eta^{n+1}\|_2^2+\|\eta^n\|_2^2),
\end{aligned}
\tag{4.8}
$$

and

$$
\begin{aligned}
&|\mathrm{Re}\left[h\sum_{j=1}^{J}(N(j,n)+N(j,n+1))\overline{(e_j^{n+1}-e_j^n)}\right. \\
&\cdot\left(\frac{F(|E(j,n+1)|^2)-F(|E(j,n)|^2)}{|E(j,n+1)|^2-|E(j,n)|^2}-\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}\right) \\
&\cdot(E(j,n+1)+E(j,n))]| \\
=&|\mathrm{Re}\left[h\sum_{j=1}^{J}(N(j,n)+N(j,n+1))\right. \\
&\cdot\left(\frac{F(|E(j,n+1)|^2)-F(|E(j,n)|^2)}{|E(j,n+1)|^2-|E(j,n)|^2}-\frac{F(|E_j^{n+1}|^2)-F(|E_j^n|^2)}{|E_j^{n+1}|^2-|E_j^n|^2}\right) \\
&\cdot\overline{(E(j,n+1)+E(j,n))}\cdot i[\frac{1}{2}((e_j^{n+1})_{x\bar{x}}+(e_j^n)_{x\bar{x}})-R^E \\
&-\frac{1}{4}(N(j,n)+N(j,n+1))\frac{F(|E(j,n+1)|^2)-F(|E(j,n)|^2)}{|E(j,n+1)|^2-|E(j,n)|^2}
\end{aligned}
$$

$$
\begin{aligned}
&\cdot (E(j,n+1) + E(j,n)) \\
&\quad + \frac{1}{4}(N_j^n + N_j^{n+1})\frac{F(|E_j^{n+1}|^2) - F(|E_j^n|^2)}{|E_j^{n+1}|^2 - |E_j^n|^2}(E_j^{n+1} + E_j^n)\Big]\Big| \\
&\leq C\tau(h^2+\tau^2)^2 + C\tau(\|e_x^{n+1}\|_2^2 + \|e_x^n\|_2^2 + \|e^{n+1}\|_2^2 + \|e^n\|_2^2 \\
&\quad + \|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2)). \qquad (4.9)
\end{aligned}
$$

It follows from (4.7), (4.8) and (4.9) that

$$
\begin{aligned}
|P_1^{n+\frac{1}{2}} - P_2^{n+\frac{1}{2}}| \leq & C\tau(h^2+\tau^2)^2 + C\tau(\|e_x^{n+1}\|_2^2 + \|e_x^n\|_2^2 + \|e^{n+1}\|_2^2 + \|e^n\|_2^2 \\
& + \|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2)). \quad \square
\end{aligned}
$$

Proof of Lemma 5. Using (3.5) and the error equation (3.3), we obtain

$$
\begin{aligned}
&|(R^E \ , \ e_j^{n+1} - e_j^n)| = |(O(h^3+\tau^3) \ , \ e_j^{n+1} - e_j^n)| \\
&+ \tau\Big|\Big(-\frac{i\tau^2}{24}E_{ttt}(j,n+\frac{1}{2}) - \frac{\tau^2}{8}E_{xxtt}(j,n+\frac{1}{2}) - \frac{\tau^2}{12}E_{xxxx}(j,n+\frac{1}{2}) \\
&+ \frac{\tau^2}{8}E(j,n+\frac{1}{2})f(|E(j,n+\frac{1}{2})|^2)N_{tt}(j,n+\frac{1}{2}) \\
&+ \frac{\tau^2}{8}N(j,n+\frac{1}{2})f(|E(j,n+\frac{1}{2})|^2)E_{tt}(j,n+\frac{1}{2}) \\
&+ \frac{\tau^2}{8}N(j,n+\frac{1}{2})E(j,n+\frac{1}{2})f'(|E(j,n+\frac{1}{2}|^2) \\
&\cdot \Big(|E(j,n+\frac{1}{2})_{tt} + \frac{\tau^2}{12}N(j,n+\frac{1}{2})E(j,n+\frac{1}{2})f''(|E(j,n+\frac{1}{2})|^2)(|E(j,n+\frac{1}{2})|^2)_t\Big)^2 \\
&\quad , \quad \frac{1}{2}((e_j^{n-1})_{x\bar{x}} + (e_j^n)_{x\bar{x}}) - O(h^2+\tau^2) \\
&- \frac{1}{4}(N(j,n) + N(j,n+1))\frac{F(|E(j,n+1)|^2) - F(|E(j,n)|^2)}{|E(j,n+1)|^2 - |E(j,n)|^2}(E(j,n+1) + E(j,n)) \\
&+ \frac{1}{4}(N_j^n + N_j^{n+1})\frac{F(|E_j^{n+1}|^2) - F(|E_j^n|^2)}{|E_j^{n+1}|^2 - |E_j^n|^2}(E_j^{n+1} + E_j^n)\Big)\Big| \\
&\leq C\tau(h^2+\tau^2)^2 + C\tau(\|e^{n+1}\|_2^2 + \|e^n\|_2^2) \\
&\quad + C\tau(\tau^4 + h^4 + \|e_x^{n+1}\|_2^2 + \|e_x^n\|_2^2 + \tau^2(h^2+\tau^2) + \tau^4 \\
&\quad + \|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2) + \|e^{n+1}\|_2^2 + \|e^n\|_2^2) \\
&\leq C\tau(h^2+\tau^2)^2 + C\tau(\|e^{n+1}\|_2^2 + \|e^n\|_2^2 + \|e_x^{n+1}\|_2^2 + \|e_x^n\|_2^2 + \|\eta^{n+1}\|_2^2 + \|\eta^n\|_2^2).
\end{aligned}
$$